XIV. *A Method of finding the Value of an infinite Series of decreasing Quantities of a certain Form, when it converges too slowly to be summed in the common Way by the mere Computation and Addition or Subtraction of some of its initial Terms.* By Francis Maseres, *Esquire, F. R. S. Cursitor Baron of the Exchequer.*

Read Feb. 13, 1777.

ARTICLE Ist. Let a, b, c, d, e, f, g, h, &c. *ad infinitum*, represent a decreasing progression of numbers, so that b shall be less than a, and c than b, and d than c, and so on of the following numbers, *ad infinitum*.

And 2dly, let these numbers be so related to each other, that they not only shall form a decreasing progression themselves, but that their differences, $a-b$, $b-c$, $c-d$, $d-e$, $e-f$, $f-g$, $g-h$, &c. shall also form a decreasing progression, so that $b-c$ shall be less than $a-b$, and $c-d$ than $b-c$, and $d-e$ than $c-d$, and so on of the following differences; and likewise, that the differences of these differences (which may be called *the second differences* of the original numbers a, b, c, d, e, f, g, h, &c. shall form a decreasing progression; and that the differences of those second differences, or *the third differences* of the original

numbers a, b, c, d, e, f, g, h, &c. ſhall alſo form a decreaſing progreſſion; and in like manner, that the differences of the ſaid third differences, or *the fourth differences*, of the original numbers a, b, c, d, e, f, g, h, &c. and the fifth and ſixth differences, and all higher differences, of the ſame numbers, ſhall alſo form decreaſing progreſſions.

And 3dly, let x be a quantity of any magnitude not greater than unity.

Upon theſe ſuppoſitions the value of the infinite ſeries $a-bx+cxx-dx^3+ex^4-fx^5+gx^6-hx^7+$ &c. (in which the ſecond, fourth, ſixth, and eighth, and every following even term, is marked with the ſign $-$, or is to be ſubtracted from that which immediately precedes it) may be determined in the following manner.

Art. 2. Compute the firſt, ſecond, third, fourth, and other ſubſequent differences of the co-efficients of the powers of x in this ſeries, that is, of the numbers b, c, d, e, f, g, h, &c. as far as ſhall be convenient. Theſe differences will be as follows.

Firſt differences, $b-c$, $c-d$, $d-e$, $e-f$, $f-g$, $g-h$, &c.

Second differences,

$b-c-\overline{c-d}$, $c-d-\overline{d-e}$, $d-e-\overline{e-f}$, $e-f-\overline{f-g}$, $f-g-\overline{g-h}$, &c.

or, $b-2c+d$, $c-2d+e$, $d-2e+f$, $e-2f+g$, $f-2g+h$, &c.

Third differences,

$b-2c+d-\overline{c-2d+e}$, $c-2d+e-\overline{d-2e+f}$, $d-2e+f-\overline{e-2f+g}$,

$e-2f+g-\overline{f-2g+h}$, &c.

or,

or, $b-3c+3d-e$, $c-3d+3e-f$, $d-3e+3f-g$, $e-3f+3g-h$, &c.

Fourth differences, $b-3c+3d-e-\overline{c-3d+3e-f}$,

$c-3d+3e-f-\overline{d-3e+3f-g}$,

$d-3e+3f-g-\overline{e-3f+3g-h}$, &c.

or, $b-4c+6d-4e+f$, $c-4d+6e-4f+g$, $d-4e+6f-4g+h$, &c.

Fifth differences, $b-4c+6d-4e+f-\overline{c-4d+6e-4f+g}$,

$c-4d+6e-4f+g-\overline{d-4e+6f-4g+h}$, &c.

or, $b-5c+10d-10e+5f-g$, $c-5d+10e-10f+5g-h$, &c.

Sixth differences,

$b-5c+10d-10e+5f-g-\overline{c-5d+10e-10f+5g-h}$, &c.

or, $b-6c+15d-20e+15f-6g+h$, &c.

Let the firſt difference of the firſt order, to wit,

$b-c$, be called D^{I};

and the firſt difference of the ſecond order, to wit,

$b-2c+d$, be called D^{II};

and the firſt difference of the third order, to wit,

$b-3c+3d-e$, be called D^{III};

and the firſt difference of the fourth order, to wit,

$b-4c+6d-4e+f$, be called D^{IV};

and the firſt difference of the fifth order, to wit,

$b-5c+10d-10e+5f-g$, be called D^{V};

and the firſt difference of the ſixth order, to wit,

$b-6c+15d-20e+15f-6g+h$, be called D^{VI};

and in like manner let the firſt differences of the ſeventh, eighth, ninth, and tenth, and every following order of differences be denoted by D^{VII}, D^{VIII}, D^{IX}, D^{X}, &c. that is, by the capital letter D, with a Roman numeral figure annexed

annexed to it, expreſſing the order of differences to which it belongs.

Theſe things being ſuppoſed, the aforeſaid infinite ſeries $a - bx + cxx - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 +$ &c. will be equal to the following differential ſeries, to wit,

$$a - \frac{bx}{1+x} - \frac{\mathrm{D}^{\mathrm{I}}xx}{\overline{1+x}|^2} - \frac{\mathrm{D}^{\mathrm{II}}x^3}{\overline{1+x}|^3} - \frac{\mathrm{D}^{\mathrm{III}}x^4}{\overline{1+x}|^4} - \frac{\mathrm{D}^{\mathrm{IV}}x^5}{\overline{1+x}|^5} - \frac{\mathrm{D}^{\mathrm{V}}x^6}{\overline{1+x}|^6} - \frac{\mathrm{D}^{\mathrm{VI}}x^7}{\overline{1+x}|^7} - \text{\&c.};$$

in which ſeries all the terms after the firſt term a are marked with the ſign $-$, or are to be ſubtracted from that term.

Art. 3. If we inſert the differences themſelves inſtead of D^{I}, D^{II}, $\mathrm{D}^{\mathrm{III}}$, D^{IV}, D^{V}, &c. in the foregoing differential ſeries (which it may perhaps ſometimes be convenient to do) that ſeries will be as follows: $a - \frac{bx}{1+x} - \overline{b-c} \times \frac{xx}{\overline{1+x}|^2}$

$$- \overline{b - 2c + d} \times \frac{x^3}{\overline{1+x}|^3} - \overline{b - 3c + 3d - e} \times \frac{x^4}{\overline{1+x}|^4}$$

$$- \overline{b - 4c + 6d - 4e + f} \times \frac{x^5}{\overline{1+x}|^5}$$

$$- \overline{b - 5c + 10d - 10e + 5f - g} \times \frac{x^6}{\overline{1+x}|^6}$$

$- \overline{b - 6c + 15d - 20e + 15f - 6g + h} \times \frac{x^7}{\overline{1+x}|^7} -$ &c. *ad infinitum.*

Of the convergency of the foregoing differential ſeries.

Art. 4. The foregoing differential ſeries will always converge with a conſiderable degree of ſwiftneſs, ſo that

that ſix or eight of its terms will give the value of the whole (and conſequently that of the original ſeries $a-bx+cxx-dx^3+ex^4-fx^5+gx^6-hx^7+$ &c. to which it is equal) exact to ſeveral places of figures, even in the moſt difficult caſes: for if x is $=1$ (which is its greateſt poſſible magnitude) $1+x$ will be $=1+1$ or 2, and conſequently $\overline{1+x}|^2$, $\overline{1+x}|^3$, $\overline{1+x}|^4$, $\overline{1+x}|^5$, and the following powers of $1+x$, will be equal to 4, 8, 16, 32, and the following powers of 2; and the powers of the fraction $\frac{x}{1+x}$ will be equal to the powers of $\frac{1}{2}$. Therefore the ſeries $a-\frac{bx}{1+x}-\frac{D^{I}xx}{\overline{1+x}|^2}-\frac{D^{II}x^3}{\overline{1+x}|^3}-\frac{D^{III}x^4}{\overline{1+x}|^4}-\frac{D^{IV}x^5}{\overline{1+x}|^5}-\frac{D^{V}x^6}{\overline{1+x}|^6}-\frac{D^{VI}x^7}{\overline{1+x}|^7}-$ &c. will in this caſe be = to $a-\frac{b}{2}-\frac{D^{I}}{4}-\frac{D^{II}}{8}-\frac{D^{III}}{16}-\frac{D^{IV}}{32}-\frac{D^{V}}{64}-\frac{D^{VI}}{128}-$ &c. the terms of which decreaſe in a greater proportion than that of 1 to 2, becauſe the numerators a, b, D^{I}, D^{II}, D^{III}, D^{IV}, D^{V}, D^{VI}, &c. form a decreaſing progreſſion, and the denominators increaſe in the proportion of 2 to 1.

Of the inveſtigation of the foregoing differential ſeries.

Art. 5. The foregoing differential ſeries was inveſtigated by firſt, ſuppoſing the original ſeries $a-bx+cxx-dx^3+ex^4-fx^5+gx^6-hx^7+$ &c. to be equal to another ſeries whoſe terms ſhould involve the ſame powers of x as the former, but in which every power of x ſhould be multiplied

multiplied into the ſame power of the fraction $\frac{1}{1+x}$, in order to accelerate their convergency, and then inquiring what would be the co-efficients of the terms of ſuch a ſeries, if ſuch a ſeries is poſſible, and what would be the ſigns to be prefixed to them, or in what manner they would be connected with the firſt term, whether by addition or ſubtraction. In order to this inquiry, I denoted the unknown co-efficients of the aſſumed ſeries by the capital letters P, Q, R, S, T, V, &c. and wrote down the terms of it near each other, without prefixing to them either of the ſigns + and −, but ſeparated them from each other only by a comma; ſo that the fundamental equation, from which I derived the differential ſeries above-mentioned, was as follows: $a-bx+cxx-dx^3+ex^4-fx^5+gx^6-hx^7+$ &c. is $=$ P, $\frac{Qx}{1+x}$, $\frac{Rxx}{\overline{1+x}|^2}$, $\frac{Sx^3}{\overline{1+x}|^3}$, $\frac{Tx^4}{\overline{1+x}|^4}$, $\frac{Vx^5}{\overline{1+x}|^5}$, &c. By neceſſary deductions from this equation it appeared that P would be equal to a; and that all the following terms, of the aſſumed ſeries, to wit, $\frac{Qx}{1+x}$, $\frac{Rxx}{\overline{1+x}|^2}$, $\frac{Sx^3}{\overline{1+x}|^3}$, $\frac{Tx^4}{\overline{1+x}|^4}$, $\frac{Vx^5}{\overline{1+x}|^5}$, &c. muſt be ſubtracted from the firſt term P, or a; and that Q would be equal to $b-c$, or D$^{\text{I}}$; and R $=b-c-\overline{c-d}$, or $b-2c+d$, or D$^{\text{II}}$; and S $=b-2c+d-\overline{c-2d+e}$, or $b-3c+3d-e$, or D$^{\text{III}}$; and

T=

T $= b-4c+6d-4e+f$, or D^{IV}; and V $= b-5c+10d-10e+5f-g$, or D^{V}; and so on of the following co-efficients, to wit, that every new co-efficient of the assumed series is equal to the first difference of the next order of the differences derived from the original co-efficients b, c, d, e, f, g, h, &c. And from hence I concluded that the series $a-bx+cxx-dx^3+ex^4-fx^5+gx^6-hx^7+$ &c. was equal to the series,

$$a-\frac{bx}{1+x}-\frac{D^{I}xx}{\overline{1+x}|^2}-\frac{D^{II}x^3}{\overline{1+x}|^3}-\frac{D^{III}x^4}{\overline{1+x}|^4}-\frac{D^{IV}x^5}{\overline{1+x}|^5}-\frac{D^{V}x^6}{\overline{1+x}|^6}-\frac{D^{VI}x^7}{\overline{1+x}|^7}-\text{ \&c.}$$

Art. 6. The thought of supposing the original series $a-bx+cxx-dx^3+ex^4-fx^5+$ &c. to be equal to the series P, $\frac{Qx}{1+x}$, $\frac{Rxx}{\overline{1+x}|^2}$, $\frac{Sx^3}{\overline{1+x}|^3}$, $\frac{Tx^4}{\overline{1+x}|^4}$, $\frac{Vx^5}{\overline{1+x}|^5}$, &c. containing the powers of x multiplied into the same powers of the fraction $\frac{1}{1+x}$ in order to accelerate their convergency, occurred to me in consequence of reading the late Mr. THOMAS SIMPSON's Mathematical Dissertations, p. 62, 63. concerning the summation of serieses, in which he makes a supposition of a similar kind. Yet there seems to be a considerable difference between his proposition and that which is the subject of these pages; for he seems to suppose his quantities p, q, r, s, t, &c. (which answer to a, b, c, d, e, &c. in the notation made use of in the above serieses) to form an increasing progression of terms, and accordingly subtracts p from q, and q from r,

 and

and r from s, and s from t, and so on; and he seems also to suppose the differences $q-p$, $r-q$, $s-r$, $t-s$, &c. to form an increasing progression, and every subsequent order of differences to form likewise an increasing progression, and accordingly subtracts $q-p$ from $r-q$, and $r-q$ from $s-r$, and $s-r$ from $t-s$, and so on; whereas in the foregoing series $a-bx+cxx-dx^3+ex^4-fx^5+gx^6-hx^7$ + &c. the numbers a, b, c, d, e, f, g, h, &c. are supposed to form a decreasing progression of terms, as they are most commonly found to do in the serieses that occur in the solution of mathematical or philosophical problems.

Examples of the usefulness of the foregoing differential series in finding the values of infinite serieses whose terms decrease very slowly.

Computations of the lengths of circular arcs by means of infinite serieses derived from their tangents.

Art. 7. It is well known, that if r be put for the radius of a circle, and t for the tangent of any arch in it that is not greater than 45°, the magnitude of the arch whose tangent is t will be expressed by the infinite series $t-\frac{t^3}{3rr}+\frac{t^5}{5r^4}-\frac{t^7}{7r^6}+\frac{t^9}{9r^8}-\frac{t^{11}}{11r^{10}}+\frac{t^{13}}{13r^{12}}-\frac{t^{15}}{15r^{14}}$ + &c. This series converges with great swiftness when the tangent is

much leſs than the radius; but when the tangent is nearly equal to the radius, it converges exceeding ſlowly; and when it is quite equal to the radius, or the arch is equal to 45°, the decreaſe of the terms is ſo ſlow as to make the computation of it in the common way, by computing the value of its initial terms, abſolutely impracticable. For Sir ISAAC NEWTON has obſerved concerning this ſeries in that extreme caſe (which then becomes equal to $r - \frac{r}{3} + \frac{r}{5} - \frac{r}{7} + \frac{r}{9} - \frac{r}{11} + \frac{r}{13} - \frac{r}{15} +$ &c.) and another ſeries that is almoſt as ſlow as this, that to exhibit its value exact to twenty decimal places of figures, there would be occaſion for no leſs than five thouſand millions of its terms, to compute which would take up above a thouſand years. See Sir ISAAC NEWTON's ſecond letter to Mr. OLDENBURGH, dated October 24, 1676, in the *Commercium Epiſtolicum*, p. 159. In theſe caſes therefore it will be convenient to make uſe of ſome artifice to diſcover the value of the ſeries $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} +$ &c.; and we ſhall find the application of the differential ſeries abovementioned to be a very proper artifice for this purpoſe.

Art. 8. In order to make this application, we muſt conſider the ſeries $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} +$ &c. as being the product of the multiplication of t into the

ſeries

ſeries $1-\frac{tt}{3rr}+\frac{t^4}{5r^4}-\frac{t^6}{7r^6}+\frac{t^8}{9r^8}-\frac{t^{10}}{11r^{10}}+\frac{t^{12}}{13r^{12}}-\frac{t^{14}}{15r^{14}}+$ &c. and muſt ſubſtitute x inſtead of $\frac{tt}{rr}$ in the terms of this laſt ſeries, by which means it will be converted into the ſeries $1-\frac{x}{3}+\frac{xx}{5}-\frac{x^3}{7}+\frac{x^4}{9}-\frac{x^5}{11}+\frac{x^6}{13}-\frac{x^7}{15}+$ &c. This ſeries is of the ſame form with the original ſeries above-mentioned, $a-bx+cxx-dx^3+ex^4-fx^5+gx^6-hx^7+$ &c. the numeral co-efficients $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{13}, \frac{1}{15}$, &c. of the powers of x in the former ſeries anſwering to the literal or general co-efficients a, b, c, d, e, f, g, h, &c. of the ſame powers in the latter ſeries. And theſe numeral co-efficients evidently form a decreaſing progreſſion, as the co-efficients a, b, c, d, e, f, g, h, &c. are ſuppoſed to do; and we ſhall find, upon examination, that the differences of theſe numeral co-efficients, of the ſeveral ſucceſſive orders, alſo conſtitute decreaſing progreſſions, as the ſeveral ſucceſſive orders of differences of the co-efficients a, b, c, d, e, f, g, h, &c. are ſuppoſed to do. Conſequently the ſeries $1-\frac{x}{3}+\frac{xx}{5}-\frac{x^3}{7}+\frac{x^4}{9}-\frac{x^5}{11}+\frac{x^6}{13}-\frac{x^7}{15}+$ &c. will be equal to the differential ſeries

$$a-\frac{bx}{1+x}-\frac{\mathrm{D^I}xx}{\overline{1+x}|^2}-\frac{\mathrm{D^{II}}x^3}{\overline{1+x}|^3}-\frac{\mathrm{D^{III}}x^4}{\overline{1+x}|^4}-\frac{\mathrm{D^{IV}}x^5}{\overline{1+x}|^5}-\frac{\mathrm{D^V}x^6}{\overline{1+x}|^6}-\frac{\mathrm{D^{VI}}x^7}{\overline{1+x}|^7}- \text{\&c. if}$$

we ſuppoſe the letters a, b, c, d, e, f, g, h, &c. to be equal to the numbers $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{13}, \frac{1}{15}$, &c. and $\mathrm{D^I}, \mathrm{D^{II}}, \mathrm{D^{III}}, \mathrm{D^{IV}}, \mathrm{D^V}, \mathrm{D^{VI}}$, &c. to be the firſt differences of

the ſeveral orders of differences of thoſe numbers, beginning from the ſecond term $\frac{1}{3}$. Now the values of theſe numbers, 1, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, $\frac{1}{11}$, $\frac{1}{13}$, $\frac{1}{15}$, &c. and of their differences of the ſeveral ſucceſſive orders, beginning from the ſecond term $\frac{1}{3}$, will, when expreſſed in decimal fractions, be as follows:

1 is	=	1.000,000,000,000;
$\frac{1}{3}$	=	.333,333,333,333;
$\frac{1}{5}$	=	.200,000,000,000;
$\frac{1}{7}$	=	.142,857,142,857;
$\frac{1}{9}$	=	.111,111,111,111;
$\frac{1}{11}$	=	.090,909,090,909;
$\frac{1}{13}$	=	.076,923,076,923;
$\frac{1}{15}$	=	.066,666,666,666.

The differences of theſe numbers, beginning from the ſecond term, .333,333,333,33, are as follows:

Firſt differences.	Second differences.
.133,333,333,333;	.076,190,476,190;
.057,142,857,143;	.025,396,825,397;
.031,746,031,746;	.011,544,011,544;
.020,202,020,202;	.006,216,006,216;
.013,986,013,986;	.003,729,603,729;
.010,256,410,257;	&c.
&c.	

Third

Third differences.

.050,793,650,793;
.013,852,813,853;
.005,328,005,328;
.002,486,402,487;
&c.

Fourth differences.

.036,940,836,940;
.008,524,808,525;
.002,841,602,841;

&c.

Fifth differences.

.028,416,028,415;
.005,683,205,684;
&c.

Sixth differences.

.022,732,822,731;

&c.

Therefore D^{I} is = .133,333,333,333;

D^{II} = .076,190,476,190;

D^{III} = .050,793,650,793;

D^{IV} = .036,940,836,940;

D^{V} = .028,416,028,415;

D^{VI} = .022,732,822,731.

Therefore the ſeries $1-\frac{x}{3}+\frac{xx}{5}-\frac{x^3}{7}+\frac{x^4}{9}-\frac{x^5}{11}+\frac{x^6}{13}-\frac{x^7}{15}+$ &c. is equal to the ſeries

$$1 - .333,333,333,333, \times \frac{x}{1+x}$$

$$- .133,333,333,333, \times \frac{xx}{\overline{1+x}|^2}$$

$$- .076,190,476,190, \times \frac{x^3}{\overline{1+x}|^3}$$

$$- .050,793,650,793, \times \frac{x^4}{\overline{1+x}|^4}$$

− .036,

$$- .036,940,836,940, \times \frac{x^5}{\overline{1+x}|^5}$$

$$- .028,416,028,415, \times \frac{x^6}{\overline{1+x}|^6}$$

$$- .022,732,822,731, \times \frac{x^7}{\overline{1+x}|^7}$$

− &c.; and consequently the product of this latter series into the tangent t will be equal to the product of the former series $1 - \frac{x}{3} + \frac{xx}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{x^5}{11} + \frac{x^6}{13} - \frac{x^7}{15} +$ &c. into the same quantity, that is, to the product of the series $1 - \frac{tt}{3rr} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \frac{t^8}{9r^8} - \frac{t^{10}}{11r^{10}} + \frac{t^{12}}{13r^{12}} - \frac{t^{14}}{15r^{14}} +$ &c. into the tangent t, or to the original series $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} +$ &c. which expresses the magnitude of the arch of which t is the tangent.

Computation of an arch of 30 *degrees.*

Art. 9. Now let t be the tangent of 30°, which is $= r \times \frac{1}{\sqrt{3}}$. Then will tt be $= \frac{rr}{3}$; and $\frac{tt}{rr}$, or x, $= \frac{rr}{3rr}$, or $\frac{1}{3}$. Therefore $1 + x$ will be $= 1 + \frac{1}{3} = \frac{3}{3} + \frac{1}{3} = \frac{4}{3}$, and $\frac{x}{1+x}$ will be $= \frac{\frac{1}{3}}{\frac{4}{3}} = \frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$. Therefore $\frac{xx}{\overline{1+x}|^2}$ will be $= \frac{1}{16}$, and $\frac{x^3}{\overline{1+x}|^3}$ $= \frac{1}{64}$, and $\frac{x^4}{\overline{1+x}|^4} = \frac{1}{256}$, and $x^5 = \frac{1}{1024}$, and $x^6 = \frac{1}{4096}$, and x^7 $= \frac{1}{16384}$. Consequently the differential series will in this

case

case be equal to

$$1 - .333,333,333,333, \times \frac{1}{4}$$
$$- .133,333,333,333, \times \frac{1}{16}$$
$$- .076,190,476,190, \times \frac{1}{64}$$
$$- .050,793,650,793, \times \frac{1}{256}$$
$$- .036,940,836,940, \times \frac{1}{1024}$$
$$- .028,416,028,415, \times \frac{1}{4096}$$
$$- .022,732,822,731, \times \frac{1}{16384}$$
$$- \&c. = 1 - .083,333,333,333,$$
$$- .008,333,333,333,$$
$$- .001,190,476,190,$$
$$- .000,198,412,698,$$
$$- .000,036,075,036,$$
$$- .000,006,937,506,$$
$$- .000,001,387,501,$$
$$- \&c.$$

$$= 1 - .093,099,955,597, = 0.906,900,044,403.$$

Therefore the series $1 - \frac{x}{3} + \frac{xx}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{x^5}{11} + \frac{x^6}{13} - \frac{x^7}{15} + \&c.$ or $1 - \frac{tt}{3rr} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \frac{t^8}{9r^8} - \frac{t^{10}}{11r^{10}} + \frac{t^{12}}{13r^{12}} - \frac{t^{14}}{15r^{14}} + \&c.$ is in this case $= 0.906,900,044,403$, or (neglecting the latter figures after the sixth place of figures, because we are sure they are

are not exact) 0.906,900. Therefore the product of the series $1 - \frac{tt}{3rr} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \frac{t^8}{9r^8} - \frac{t^{10}}{11r^{10}} + \frac{t^{12}}{13r^{12}} - \frac{t^{14}}{15r^{14}} +$ &c. into the tangent t is equal to $0.906,900, \times t = 0.906,900 \times r \times \frac{1}{\sqrt{3}}$

$$= 0.906,900, \times r \times \frac{1}{1.732,050,8} = 0.906,900 \times r$$

$\times .577,350,2 = 0.523,598,8 \times r$; that is, the series $t - \frac{t^3}{3rr} + \frac{t^5}{5r^5} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} +$ &c. (which expresses the magnitude of the arch of which t is the tangent) is in this case $= 0.523,598,8 \times r$, or an arch of $30°$ is equal to $0.523,598,8 \times r$.

Art. 10. This value of an arch of $30°$ is exact in the six first places of figures, and errs only an unit in the seventh figure, which should be a 7 instead of an 8, the more exact value of that arch being 0.523,598,775,598, &c. And thus by the help of only eight terms of the differential series

$$a - \frac{bx}{1+x} - \frac{D^{I}xx}{\overline{1+x}|^2} - \frac{D^{II}x^3}{\overline{1+x}|^3} - \frac{D^{III}x^4}{\overline{1+x}|^4} - \frac{D^{IV}x^5}{\overline{1+x}|^5} - \frac{D^{V}x^6}{\overline{1+x}|^5} - \frac{D^{VI}x^7}{\overline{1+x}|^7} - \text{\&c.}$$

we have obtained the value of the series

$t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} +$ &c. in the case of an arch of 30 degrees, exact to six places of figures. This degree of exactness is the same with that which we should attain by computing twelve terms of the series $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} +$ &c. itself, as will appear from the following calculation.

Art. 11. The series $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} + \frac{t^{17}}{17r^{16}} - \frac{t^{19}}{19r^{18}} + \frac{t^{21}}{21r^{20}} - \frac{t^{23}}{23r^{22}}$ + &c. is $= t \times$ the series $1 - \frac{tt}{3rr} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \frac{t^8}{9r^8} - \frac{t^{10}}{11r^{10}} + \frac{t^{12}}{13r^{12}} - \frac{t^{14}}{15r^{14}} + \frac{t^{16}}{17r^{16}} - \frac{t^{18}}{19r^{18}} + \frac{t^{20}}{21r^{20}} - \frac{t^{22}}{23r^{22}}$ + &c. =, in the case of an arch of 30°, to $r \times \frac{1}{\sqrt{3}}$ into the series $1 - \frac{1}{3 \times 3} + \frac{1}{5 \times 9} - \frac{1}{7 \times 27} + \frac{1}{9 \times 81} - \frac{1}{11 \times 243} + \frac{1}{13 \times 729} - \frac{1}{15 \times 2187} + \frac{1}{17 \times 6561} - \frac{1}{19 \times 19683} + \frac{1}{21 \times 59049} - \frac{1}{23 \times 177,147}$ + &c.

$$= r \times \frac{1}{\sqrt{3}} \times \text{the series } 1.000,000,000,000, - \frac{.333,333,333,333,}{3}$$

$$+ \frac{.111,111,111,111,}{5} - \frac{.037,037,037,037,}{7}$$

$$+ \frac{.012,345,679,012,}{9} - \frac{.004,115,226,337,}{11}$$

$$+ \frac{.001,371,742,112,}{13} - \frac{.000,457,247,370,}{15}$$

$$+ \frac{.000,152,415,790,}{17} - \frac{.000,050,805,263,}{19}$$

$$+ \frac{.000,016,935,087,}{21} - \frac{.000,005,645,029,}{23}$$

+ &c.

$= r \times \frac{1}{\sqrt{3}} \times$ the series

1.000,000,000,000, − .111,111,111,111,
+ .022,222,222,222, − .005,291,005,291,
+ .001,371,742,112, − .000,374,111,485,
+ .000,105,518,624, − .000,030,483,158,
+ .000,008,965,634, − .000,002,673,961,
+ .000,000,806,432, − .000,000,245,436,
+ &c.

I

=

$= r \times \frac{1}{\sqrt{3}} \times \overline{1.023,709,255,024, - .116,809,630,442,}$
$+$ &c.

$= r \times \frac{1}{\sqrt{3}} \times .906,899,624,582,$

$= r \times \frac{1}{1.732,050,8} \times .906,899,624,582,$

$= r \times .577,350,2 \times .906,899,624,582,$ = (if we neglect the six latter figures of 906,899,624,582, which we know to be not exact) $r \times .577,350,2 \times 906,899,$ $= r \times .523,598,319,029,8$; of which the first six figures .523,598, are exact.

Computation of an arch of 45 degrees.

Art. 12. Now let the tangent t be equal to the radius r, or the arch (whose magnitude is expressed by the series $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} +$ &c.) be an arch of 45°. This series will, in this case, become equal to $r - \frac{r}{3} + \frac{r}{5} - \frac{r}{7} + \frac{r}{9} - \frac{r}{11} + \frac{r}{13} - \frac{r}{15} +$ &c. of which the first eight terms will give the value of the whole exact to only one figure, as will appear by the following computation. These terms are equal to $r \times$ the eight terms $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15}$, that is, to $r \times$ the eight terms

$$1.000,000,000,000, - .333,333,333,333,$$
$$+ .200,000,000,000, - .142,857,142,857,$$
$$+ .111,111,111,111, - .090,909,090,909,$$
$$+ .076,923,076,923, - .066,666,666,666,$$

 =

$=r\times\sqrt{1.388,034,187,034,-.633,766,233,765,}$ $=r\times.754,267,943,269$; which agrees with the value of the whole ſeries $r-\frac{r}{3}+\frac{r}{5}-\frac{r}{7}+\frac{r}{9}-\frac{r}{11}+$ &c. only in the higheſt figure 7, the more exact value of that ſeries being .785,398,163,397, &c. But, if we compute eight terms of the differential ſeries which is equal to the ſeries $r-\frac{r}{3}+\frac{r}{5}-\frac{r}{7}+\frac{r}{9}-\frac{r}{11}+$ &c., we ſhall thereby obtain its value exact to three places of figures; which is as great a degree of exactneſs as would be attained by computing about five hundred terms of the ſeries $r-\frac{r}{3}+\frac{r}{5}-\frac{r}{7}+\frac{r}{9}-\frac{r}{11}+\frac{r}{13}-\frac{r}{15}+$ &c. itſelf. The computation of the eight firſt terms of the ſaid differential ſeries is as follows.

Art. 13. Since t is in this caſe $=r$, tt will be $=rr$, and conſequently $\frac{tt}{rr}$, or x, will be $=1$. Therefore xx, x^3, x^4, x^5, and all the other powers of x, will in this caſe be equal to 1, and $1+x$ will be equal to $1+1$, or 2, and the powers of $1+x$ to the powers of 2. Therefore the fraction $\frac{x}{1+x}$ and its powers will be equal in this caſe to the fraction $\frac{1}{2}$ and its powers. Therefore the general differential ſeries in art. 8. to wit,

$$1 - .333,333,333,333, \times \frac{x}{1+x}$$
$$- .133,333,333,333, \times \frac{xx}{\overline{1+x}|^2}$$
$$- .076,190,476,190, \times \frac{x^3}{\overline{1+x}|^3}$$
$$- .050,793,650,793, \times \frac{x^4}{\overline{1+x}|^4}$$
$$- .036,940,836,940, \times \frac{x^5}{\overline{1+x}|^5}$$
$$- .028,416,028,415, \times \frac{x^6}{\overline{1+x}|^6}$$
$$- .022,732,822,731, \times \frac{x^7}{\overline{1+x}|^7}$$

– &c. will become in this caſe equal to

$$1 - .333,333,333,333, \times \frac{1}{2}$$
$$- .133,333,333,333, \times \frac{1}{4}$$
$$- .076,190,476,190, \times \frac{1}{8}$$
$$- .050,793,650,793, \times \frac{1}{16}$$
$$- .036,940,836,940, \times \frac{1}{32}$$
$$- .028,416,028,415, \times \frac{1}{64}$$
$$- .022,732,822,731, \times \frac{1}{128}$$
$$- \&c. = 1 - .166,666,666,666,$$
$$- .033,333,333,333,$$
$$- .009,523,809,523,$$

$$\begin{aligned} &- .003,174,603,174, \\ &- .001,154,401,154, \\ &- .000,444,000,443, \\ &- .000,177,600,177, \\ &- \&c. \end{aligned}$$

$$= 1 - .214,474,414,470, = .785,525,585,530.$$

Therefore the series $1 - \frac{x}{3} + \frac{xx}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{x^5}{11} + \frac{x^6}{13} - \frac{x^7}{15} + \&c.$ or $1 - \frac{tt}{3rr} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \frac{t^8}{9r^8} - \frac{t^{10}}{11r^{10}} + \frac{t^{12}}{13r^{12}} - \frac{t^{14}}{15r^{14}} + \&c.$ is equal to .785,525,585,530; and consequently the series $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} + \&c.$ is in this case $= t \times .785,525,585,530, = r \times .785,525,585,530$; that is, the length of an arch of 45°, in a circle whose radius is r, is $= r \times .785,525,585,530$; which number is true to three places of figures, the more exact value of that arch being $r \times .785,398,163,397$, &c.

Art. 14. It has been asserted in art. 12, that in order to obtain the value of the series $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} + \&c.$ exact to 3 places of decimal figures by the mere computation of its terms, in the case of an arch of 45°, we must compute at least 500 of its terms. This may be proved in the following manner. The indexes of the powers of t in that series are the odd numbers 1, 3, 5, 7, 9, 11, 13, 15, &c.

&c. in their natural order; to which if we add an unit, the numbers thereby produced will be the even numbers 2, 4, 6, 8, 10, 12, 14, 16, &c. in their natural order, which are the doubles of the natural numbers, 1, 2, 3, 4, 5, 6, 7, &c. Therefore the number of terms of that ſeries from the beginning of it to any given term in it, including the ſaid term, is always half the number that is produced by adding an unit to the index of t in the ſaid term. Thus, if we take the term $\frac{t^{11}}{11r^{10}}$, and add 1 to 11, which is the index of the power of t in it, the ſum will be 12, the half of which is 6, which is the number of terms in the ſeries from the beginning of it to the term $\frac{t^{11}}{11r^{10}}$, including the ſaid term, that term being the ſixth term in the ſeries. If therefore we take the term $\frac{t^{999}}{999r^{998}}$, and are deſirous of knowing its place in the ſeries, or the number of terms from the beginning of the ſeries to that term incluſively, we muſt add 1 to the index of the power of t in its numerator, which will increaſe it to 1000; and half this ſum, to wit, 500, will be the number of terms from the beginning of the ſeries to the term $\frac{t^{999}}{999r^{998}}$ incluſively; or, in other words, this term will be the 500th term of the ſeries. To arrive therefore at thoſe terms of the ſeries in which the indexes

dexes of the powers of t are greater than 999, or 1000, or in which the numeral co-efficients of the terms (which, by the law of this ſeries, are equal to 1 divided by theſe indexes) are leſs than $\frac{1}{999}$ or $\frac{1}{1000}$, it is neceſſary to compute 500 of its terms. Now when t is $= r$, and conſequently the literal parts of the terms of this ſeries do not converge at all, it is evidently neceſſary to carry the computation as far as thoſe terms in which the numeral co-efficients of the terms are leſs than $\frac{1}{999}$ or $\frac{1}{1000}$, in order to get the value of the ſeries exact to the $\frac{1}{999}$th or $\frac{1}{1000}$th part of the radius r, or to the place of thouſandths, or the third place of decimal figures. Therefore, when t is $= r$, or the arch is $= 45°$, it is neceſſary to compute at leaſt 500 terms of the ſeries $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} +$ &c., in order to obtain the value of it exact to three places of decimal figures, that is, to the ſame degree of exactneſs to which we attained in art. 13. by computing only eight terms of the above-mentioned differential ſeries. *Q. E. D.*

Art. 15. But the beſt way of applying the aforeſaid differential ſeries to the inveſtigation of the value of one of theſe very ſlow ſerieſes, is to compute a moderate number of the firſt terms of the ſlow ſeries in the common way, and then apply the differential ſeries to the

computation of its remaining terms. The advantage of this method of proceeding will be manifest, if we apply it to the foregoing example of the series $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} +$ &c. in the case of an arch of 45°.

Compute therefore the first twelve terms of this series in the common way. These terms will be as follows:

$$t = r = r \times 1.000,000,000,000;$$

$$\frac{t^3}{3rr} = \frac{r^3}{3rr} = \frac{r}{3} = r \times .333,333,333,333;$$

$$\frac{t^5}{5r^4} = \frac{r^5}{5r^4} = \frac{r}{5} = r \times .200,000,000,000;$$

$$\frac{t^7}{7r^6} = \frac{r^7}{7r^6} = \frac{r}{7} = r \times .142,857,142,857;$$

$$\frac{t^9}{9r^8} = \frac{r^9}{9r^8} = \frac{r}{9} = r \times .111,111,111,111;$$

$$\frac{t^{11}}{11r^{10}} = \frac{r^{11}}{11r^{10}} = \frac{r}{11} = r \times .090,909,090,909;$$

$$\frac{t^{13}}{13r^{12}} = \frac{r^{13}}{13r^{12}} = \frac{r}{13} = r \times .076,923,076,923;$$

$$\frac{t^{15}}{15r^{14}} = \frac{r^{15}}{15r^{14}} = \frac{r}{15} = r \times .066,666,666,666;$$

$$\frac{t^{17}}{17r^{16}} = \frac{r^{17}}{17r^{16}} = \frac{r}{17} = r \times .058,823,529,411;$$

$$\frac{t^{19}}{19r^{18}} = \frac{r^{19}}{19r^{18}} = \frac{r}{19} = r \times .052,631,578,947;$$

$$\frac{t^{21}}{21r^{20}} = \frac{r^{21}}{21r^{20}} = \frac{r}{21} = r \times .047,619,047,619;$$

$$\frac{t^{23}}{23r^{22}} = \frac{r^{23}}{23r^{22}} = \frac{r}{23} = r \times .043,478,260,869.$$

Therefore the twelve terms $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}}$ $+ \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} + \frac{t^{17}}{17r^{16}} - \frac{t^{19}}{19r^{18}} + \frac{t^{21}}{21r^{20}} - \frac{t^{23}}{23r^{22}}$ are

$$= \left\{ \begin{array}{l} r \times 1.000,000,000,000, -r \times .333,333,333,333, \\ +r \times .200,000,000,000, -r \times .142,857,142,857, \\ +r \times .111,111,111,111, -r \times .090,909,090,909, \\ +r \times .076,923,076,923, -r \times .066,666,666,666, \\ +r \times .058,823,529,411, -r \times .052,631,578,947, \\ +r \times .047,619,047,619, -r \times .043,478,260,869, \end{array} \right\}$$

$= r \times 1.494,476,765,064, - r \times .729,876,073,581,$

$= r \times .764,600,691,483.$

Having thus found the value of the first twelve terms of the series $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} +$ &c. to be $r \times .764,600,691,483$, we must apply the differential series to the discovery of the value of the remaining part of this series, which is the series

$\frac{t^{25}}{25r^{24}} - \frac{t^{27}}{27r^{26}} + \frac{t^{29}}{29r^{28}} - \frac{t^{31}}{31r^{30}} + \frac{t^{33}}{33r^{32}} - \frac{t^{35}}{35r^{34}} + \frac{t^{37}}{37r^{36}} - \frac{t^{39}}{39r^{38}} +$ &c. *ad infinitum*. Now this series is equal to the product of $\frac{t^{25}}{r^{24}}$ into the series $\frac{1}{25} - \frac{tt}{27rr} + \frac{t^4}{29r^4} - \frac{t^6}{31r^6} + \frac{t^8}{33r^8} - \frac{t^{10}}{35r^{10}} + \frac{t^{12}}{37r^{12}} - \frac{t^{14}}{39r^{14}} +$ &c. or (putting x, as before, $= \frac{tt}{rr}$) to the product of $\frac{t^{25}}{r^{24}}$ into the

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series $\frac{1}{25} - \frac{x}{27} + \frac{xx}{29} - \frac{x^3}{31} + \frac{x^4}{33} - \frac{x^5}{35} + \frac{x^6}{37} - \frac{x^7}{39} +$ &c. which is of the same form with the series $a - bx + cxx - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 +$ &c. Therefore, if we put $a = \frac{1}{25}$, $b = \frac{1}{27}$, $c = \frac{1}{29}$, $d = \frac{1}{31}$, $e = \frac{1}{33}$, $f = \frac{1}{35}$, $g = \frac{1}{37}$, $h = \frac{1}{39}$, and so on, and compute the differential series

$$a - \frac{bx}{1+x} - \frac{D^{I}xx}{\overline{1+x}|^2} - \frac{D^{II}x^3}{\overline{1+x}|^3} - \frac{D^{III}x^4}{\overline{1+x}|^4} - \frac{D^{IV}x^5}{\overline{1+x}|^5} - \frac{D^{V}x^6}{\overline{1+x}|^6} - \frac{D^{VI}x^7}{\overline{1+x}|^7} - \text{\&c.}$$

thence resulting, the number thereby obtained will be the value of the series $\frac{1}{25} - \frac{x}{27} + \frac{xx}{29} - \frac{x^3}{31} + \frac{x^4}{33} - \frac{x^5}{35} + \frac{x^6}{37} - \frac{x^7}{39} +$ &c. or $\frac{1}{25} - \frac{tt}{27rr} + \frac{t^4}{29r^4} - \frac{t^6}{31r^6} + \frac{t^8}{33r^8} - \frac{t^{10}}{35r^{10}} + \frac{t^{12}}{37r^{12}} - \frac{t^{14}}{39r^{14}} +$ &c. This computation is as follows:

Here a is $= \frac{1}{25} = .040,000,000,000$;

$b = \frac{1}{27} = .037,037,037,037$;

$c = \frac{1}{29} = .034,482,758,620$;

$d = \frac{1}{31} = .032,258,064,516$;

$e = \frac{1}{33} = .030,303,030,303$;

$f = \frac{1}{35} = .028,571,428,571$;

$g = \frac{1}{37} = .027,027,027,027$;

$h = \frac{1}{39} = .025,641,025,641$.

 The

The differences of these numbers (beginning from the second number $\frac{1}{27}$, or .037,037,037,037,) are as follows:

First differences.

.002,554,278,416;
.002,224,694,104;
.001,955,034,413;
.001,731,601,731;
.001,544,401,544;
.001,386,001,386.

Second differences.

.000,329,584,311;
.000,269,659,891;
.000,223,432,481;
.000,187,200,187;
.000,158,400,158.

Third differences.

.000,059,924,420;
.000,046,227,409;
.000,036,232,294;
.000,028,800,028.

Fourth differences.

.000,013,697,010;
.000,009,995,115;
.000,007,432,265.

Fifth differences.

.000,003,701,894;
.000,002,562,850.

Sixth differences.

.000,001,139,044.

Therefore D^{I} is = .002,554,278,416;
D^{II} = .000,329,584,311;
D^{III} = .000,059,924,420;
D^{IV} = .000,013,697,010;
D^{V} = .000,003,701,894;
D^{VI} = .000,001,139,044.

Consequently the differential series

$$a-\frac{bx}{1+x}-\frac{D^{I}xx}{\overline{1+x}|^{2}}-\frac{D^{II}x^{3}}{\overline{1+x}|^{3}}-\frac{D^{III}x^{4}}{\overline{1+x}|^{4}}-\frac{D^{IV}x^{5}}{\overline{1+x}|^{5}}-\frac{D^{V}x^{6}}{\overline{1+x}|^{6}}-\frac{D^{VI}x^{7}}{\overline{1+x}|^{7}}-\&c. \text{ is}=\text{to}$$

.040,

$$
\begin{aligned}
& .040,000,000,000, \\
- & .037,037,037,037, \times \frac{x}{1+x} \\
- & .002,554,278,416, \times \frac{xx}{\overline{1+x}|^2} \\
- & .000,329,584,311, \times \frac{x^3}{\overline{1+x}|^3} \\
- & .000,059,924,420, \times \frac{x^4}{\overline{1+x}|^4} \\
- & .000,013,697,010, \times \frac{x^5}{\overline{1+x}|^5} \\
- & .000,003,701,894, \times \frac{x^6}{\overline{1+x}|^6} \\
- & .000,001,139,044, \times \frac{x^7}{\overline{1+x}|^7} \\
- & \&c.
\end{aligned}
$$

But, since t is in this case $=r$, $\frac{tt}{rr}$, or x, will be $=1$, and consequently $\frac{x}{1+x}$ will be $=\frac{1}{1+1}$ or $\frac{1}{2}$. Therefore the foregoing differential series is in this case equal to

$$
\begin{aligned}
& .040,000,000,000, \\
- & .037,037,037,037, \times \frac{1}{2} \\
- & .002,554,278,416, \times \frac{1}{4} \\
- & .000,329,584,311, \times \frac{1}{8} \\
- & .000,059,924,420, \times \frac{1}{16} \\
- & .000,013,697,010, \times \frac{1}{32}
\end{aligned}
$$

— .000,

$$- .000,003,701,894, \times \frac{1}{64}$$

$$- .000,001,139,044, \times \frac{1}{128}$$

$$\begin{aligned} - \&c. = .040,000,000,000, &- .018,518,518,518, \\ &- .000,638,569,604, \\ &- .000,041,198,038, \\ &- .000,003,745,276, \\ &- .000,000,428,031, \\ &- .000,000,057,842, \\ &- .000,000,008,898, \\ &- \&c. \end{aligned}$$

$$= .040,000,000,000, - .019,202,526,207, - \&c.$$

$$= .020,797,473,793 - \&c.$$

Therefore the ſeries $\frac{1}{25} - \frac{x}{27} + \frac{xx}{29} - \frac{x^3}{31} + \frac{x^4}{33} - \frac{x^5}{35} + \frac{x^6}{37} - \frac{x^7}{39} +$ &c. or $\frac{1}{25} - \frac{tt}{27rr} + \frac{t^4}{29r^4} - \frac{t^6}{31r^6} + \frac{t^8}{33r^8} - \frac{t^{10}}{35r^{10}} + \frac{t^{12}}{37r^{12}} - \frac{t^{14}}{39r^{14}} +$ &c. is in this caſe $= .020,797,473,793$. Therefore the ſeries $\frac{t^{25}}{25r^{24}} - \frac{t^{27}}{27r^{26}} + \frac{t^{29}}{29r^{28}} - \frac{t^{31}}{31r^{30}} + \frac{t^{33}}{33r^{32}} - \frac{t^{35}}{35r^{34}} + \frac{t^{37}}{37r^{36}} - \frac{t^{39}}{39r^{38}} +$ &c. is in this caſe equal to $\frac{t^{25}}{r^{24}} \times .020,797,473,793$, that is, to $\frac{r^{25}}{r^{24}} \times .020,797,473,793$, or $r \times .020,797,473,793$; that is, the remainder of the infinite ſeries $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} +$ &c. after the firſt twelve terms, is $= r \times .020,797,473,793$. But we before found

 thoſe

thoſe firſt twelve terms to be $= r \times .764,600,691,483$. Therefore the whole ſeries $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} +$ &c. *ad infinitum* is in this caſe $= r \times .764,600,691,483,$

$$+ r \times .020,797,473,793, = r \times .785,398,165,276,$$

which is true to eight places of figures, the more exact value of that ſeries being $r \times .785,398,163,397$, &c.; ſo that the value here found for this ſeries, by the help of only eight terms of the differential ſeries, differs from its true value by leſs than an unit in the eighth place of decimal figures, that is, by leſs than an hundred-millionth part of the radius r, which is a degree of exactneſs that could not have been attained by the mere computation of the ſeries $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} +$ &c. itſelf without computing fifty millions of its terms. There cannot be a ſtronger inſtance of the utility of that differential ſeries.

Computation of the ſeries which expreſſes the time of the deſcent of a pendulum through the arch of a circle.

Art. 16. As another example of the utility of the foregoing differential ſeries in finding the value of a ſeries that converges very ſlowly, I will now apply it to the ſeries which expreſſes the time of deſcent of a heavy body through a circular arch of 90°, which de-

creaſes

creases almost as slowly as the above-mentioned series $r - \frac{r}{3} + \frac{r}{5} - \frac{r}{7} + \frac{r}{9} - \frac{r}{11} + \frac{r}{13} - \frac{r}{15} +$ &c., which expresses the magnitude of a circular arch of 45° in a circle whose radius is r.

Art. 17. If a heavy body, or a pendulum, be supposed to descend by the mere force of gravity through any arch of a circle not exceeding the arch of a quadrant, or 90°; and the motion be supposed to begin from a state of rest, and to continue till the bob of the pendulum, or the heavy body, comes to the lowest point of the circle; and the radius of the circle be called r, the perpendicular height, or versed sine, of the arch through which the descent is made, be called v, and the right sine of the same arch be called s; and π be put for the number 1.570,796,326,794, &c. which expresses the semi-circumference of a circle whose diameter is called 1; and the time of the fall of a heavy body through the versed sine v, or the perpendicular altitude of the arch through which the pendulum descends, be denoted by v; the time of the descent of the pendulum through the said circular arch, corresponding to the versed sine or altitude v, to the lowest point of the circle, will be expressed by the product of $\pi \times \frac{rv}{s}$ into the series

$$1 - \frac{1.1}{2.2} \times \frac{vv}{ss} + \frac{1.1.3.3}{2.2.4.4} \times \frac{v^4}{s^4} - \frac{1.1.3.3.5.5}{2.2.4.4.6.6} \times \frac{v^6}{s^6} + \frac{1.1.3.3.5.5.7.7}{2.2.4.4.6.6.8.8} \times \frac{v^8}{s^8}$$

—

$-\frac{1.1.3.3.5.5.7.7.9.9}{2.2.4.4.6.6.8.8.10.10} \times \frac{v^{10}}{s^{10}}$ + &c. in which the law of the continuation of the terms is very manifest, every new term being derived from the preceding term by multiplying it into the fraction $\frac{vv}{ss}$, and likewise into a numeral fraction, whose denominator is the square of the index of the powers of v and s in the new term, and whose numerator is the square of the odd number that is less than the said index by an unit.

Art. 18. Let the numeral co-efficients of the terms of this series be denoted by the capital letters of the alphabet, A, B, C, D, E, F, G, H, &c. in their natural order, so that A shall be equal to 1, and B shall be = $\frac{1.1}{2.2}$, and $C = \frac{1.1.3.3}{2.2.4.4}$, and $D = \frac{1.1.3.3.5.5}{2.2.4.4.6.6}$, and so on of the rest. And we shall have

$B = \frac{1.1}{2.2}A$, $C = \frac{3.3}{4.4}B$, $D = \frac{5.5}{6.6}C$, $E = \frac{7.7}{8.8}D$, $F = \frac{9.9}{10.10}E$, $G = \frac{11.11}{12.12}F$, $H = \frac{13.13}{14.14}G$, and so on; and consequently the series

$$1 - \frac{1.1}{2.2} \times \frac{vv}{ss} + \frac{1.1.3.3}{2.2.4.4} \times \frac{v^4}{s^4} - \frac{1.1.3.3.5.5}{2.2.4.4.6.6} \times \frac{v^6}{s^6} + \text{\&c.}$$

or $A - \frac{Bvv}{ss} + \frac{Cv^4}{s^4} - \frac{Dv^6}{s^6} + \frac{Ev^8}{s^8} - \frac{Fv^{10}}{s^{10}} + \frac{Gv^{12}}{s^{12}} - \frac{Hv^{14}}{s^{14}}$ + &c. will be $A - \frac{1.1.Avv}{2.2.\ ss} + \frac{3.3.Bv^4}{4.4.\ s^4} - \frac{5.5.Cv^6}{6.6.\ s^6} + \frac{7.7\ Dv^8}{8.8.\ s^8} - \frac{9.9.\ Ev^{10}}{10.10.\ s^{10}} + \frac{11.11.Fv^{12}}{12.12.\ s^{12}} - \frac{13.13.Gv^{14}}{14.14.\ s^{14}}$ + &c. or $A - \frac{1Avv}{4ss} + \frac{9Bv^4}{16s^4} - \frac{25Cv^6}{36s^6} + \frac{49Dv^8}{64s^8} - \frac{81Ev^{10}}{100s^{10}} + \frac{121Fv^{12}}{144s^{12}} - \frac{169Gv^{14}}{196s^{14}}$ + &c. or, if we convert the co-efficients of the terms into decimal fractions,

$1 - .250,000,000,000, \times \frac{vv}{ss} + .140,625,000,000, \times \frac{v^4}{s^4}$
$- .097,656,250,000, \times \frac{v^6}{s^6} + .074,768,066,406, \times \frac{v^8}{s^8}$
$- .060,562,133,788, \times \frac{v^{10}}{s^{10}} + .050,889,015,196, \times \frac{v^{12}}{s^{12}}$
$- .043,878,793,714, \times \frac{v^{14}}{s^{14}}$ + &c. The co-efficients of these terms decreaſe ſo ſlowly (eſpecially after the firſt twelve or fourteen terms) that, when the verſed ſine v is very nearly equal to the right ſine s (as is the caſe when the arch through which the heavy body deſcends is nearly equal to 90°, or the arch of a whole quadrant of a circle) it would be neceſſary to compute a vaſt number of the terms of the ſeries in order to obtain its value exact to ſeven or eight places of figures; and, when v is quite equal to s (as is the caſe when the arch, through which the deſcent is made, is exactly equal to 90°) the computation of the value of the ſeries to that degree of exactneſs in that direct manner becomes wholly impracticable. But by the help of the differential ſeries abovementioned its value may be found, even in this caſe, to that degree of exactneſs without much difficulty; more eſpecially if we compute the firſt twelve terms of the ſeries in the common way, and then apply the differential ſeries to the inveſtigation of the remaining part of it in the ſame manner as in the laſt example. This we ſhall now proceed to do.

Art. 19. The co-efficients of the first twelve terms of the series $A - \frac{1Avv}{4ss} + \frac{9Bv^4}{16s^4} - \frac{25Cv^6}{36s^6} + \frac{49Dv^8}{64s^8} - \frac{81Ev^{10}}{100s^{10}} + \frac{121Fv^{12}}{144s^{12}} - \frac{169Gv^{14}}{196s^{14}} +$ &c. are as follows:

$$A = 1 = 1.000,000,000,000;$$

$$B = \frac{1}{4} = .250,000,000,000;$$

$$C = \frac{9B}{16} = .140,625,000,000;$$

$$D = \frac{25C}{36} = .097,656,250,000;$$

$$E = \frac{49D}{64} = .074,768,066,406;$$

$$F = \frac{81E}{100} = .060,562,133,788;$$

$$G = \frac{121F}{144} = .050,889,015,196;$$

$$H = \frac{169G}{196} = .043,878,793,714;$$

$$I = \frac{225H}{256} = .038,565,346,037;$$

$$K = \frac{289I}{324} = .034,399,336,434;$$

$$L = \frac{361K}{400} = .031,045,401,131;$$

$$M = \frac{441L}{484} = .028,287,235,328.$$

But when v is $= s$, as it is in the case of an arch of 90°, $\frac{vv}{ss}$ and all its powers will be $= 1$, and the twelve terms $A - \frac{Bvv}{ss} + \frac{Cv^4}{s^4} - \frac{Dv^6}{s^6} + \frac{Ev^8}{s^8} - \frac{Fv^{10}}{s^{10}} + \frac{Gv^{12}}{s^{12}} - \frac{Hv^{14}}{s^{14}} + \frac{Iv^{16}}{s^{16}} - \frac{Kv^{18}}{s^{18}} + \frac{Lv^{20}}{s^{20}} - \frac{Mv^{22}}{s^{22}}$

 will

will be equal to their co-efficients A−B+C−D+E−F+G−H+I−K+L−M. Therefore in this caſe the firſt twelve terms of this ſeries are

1.000,000,000,000, −.250,000,000,000,
+ .140,625,000,000, −.097,656,250,000,
+ .074,768,066,406, −.060,562,133,788,
+ .050,889,015,196, −.043,878,793,714,
+ .038,565,346,037, −.034,399,336,434,
+ .031,045,401,131, −.028,287,235,328,

which are = 1.335,892,828,770, −.514,783,749,264,
= .821,109,079,506.

Art. 20. The remaining part of this ſeries is

$$\frac{Nv^{24}}{s^{24}}-\frac{Ov^{26}}{s^{26}}+\frac{Pv^{28}}{s^{28}}-\frac{Qv^{30}}{s^{30}}+\frac{Rv^{32}}{s^{32}}-\frac{Sv^{34}}{s^{34}}+\frac{Tv^{36}}{s^{36}}-\frac{Vv^{38}}{s^{38}}+\&c.$$

$$\text{or } \frac{23.23.Mv^{24}}{24.24.s^{24}}-\frac{25.25.Nv^{26}}{26.26.s^{26}}+\frac{27.27.Ov^{28}}{28.28.s^{28}}-\frac{29.29.Pv^{30}}{30.30.s^{30}}+\frac{31.31.Qv^{32}}{32.32.s^{32}}$$

$$-\frac{33.33.Rv^{34}}{34.34.s^{34}}+\frac{35.35.Sv^{36}}{36.36.s^{36}}-\frac{37.37.Tv^{38}}{38.38.s^{38}}+\&c. \text{ or}$$

$$.025{,}979{,}075{,}500, \times \frac{v^{24}}{s^{24}} - .024{,}019{,}115{,}661 \times \frac{v^{26}}{s^{26}}$$

$$+ .022{,}334{,}101{,}169, \times \frac{v^{28}}{s^{28}} - .020{,}869{,}976{,}759 \times \frac{v^{30}}{s^{30}}$$

$$+ .019{,}585{,}984{,}048, \times \frac{v^{32}}{s^{32}} - .018{,}450{,}810{,}232, \times \frac{v^{34}}{s^{34}}$$

$$+ .017{,}440{,}001{,}955, \times \frac{v^{36}}{s^{36}} - .016{,}534{,}184{,}679, \times \frac{v^{38}}{s^{38}}$$

+ &c.; which is $= \frac{v^{24}}{s^{24}} \times$ the ſeries

$$.025{,}979{,}075{,}500, - .024{,}019{,}115{,}661, \frac{vv}{ss}$$

+ .022,

$+ .022,334,101,169, \frac{v^4}{s^4} - .020,869,976,759, \frac{v^6}{s^6}$

$+ .019,585,984,048, \frac{v^8}{s^8} - .018,450,810,232, \frac{v^{10}}{s^{10}}$

$+ .017,440,001,955, \frac{v^{12}}{s^{12}} - .016,534,184,679, \frac{v^{14}}{s^{14}}$

+ &c. or, if we fubftitute x in this laft feries inftead of $\frac{vv}{ss}$, $= \frac{v^{24}}{s^{24}} \times$ the feries

$$.025,979,075,500, - .024,019,115,661, x$$
$$+ .022,334,101,169, xx - .020,869,976,759, x^3$$
$$+ .019,585,984,048, x^4 - .018,450,810,232, x^5$$
$$+ .017,440,001,955, x^6 - .016,534,184,679, x^7$$

+ &c. Now the value of this laft feries may be difcovered by the application of the differential feries

$$a - \frac{bx}{1+x} - \frac{D^{I}xx}{\overline{1+x}|^2} - \frac{D^{II}x^3}{\overline{1+x}|^3} - \frac{D^{III}x^4}{\overline{1+x}|^4} - \frac{D^{IV}x^5}{\overline{1+x}|^5} - \frac{D^{V}x^6}{\overline{1+x}|^6} - \frac{D^{VI}x^7}{\overline{1+x}|^7} - \&c.$$

in the manner following:

Here a is $= .025,979,075,500$;
b $= .024,019,115,661$;
c $= .022,334,101,169$;
d $= .020,869,976,759$;
e $= .019,585,984,048$;
f $= .018,450,810,232$;
g $= .017,440,001,955$;
and h $= .016,534,184,679$.

Therefore

Therefore the differences of b, c, d, e, f, g, and h, of the several successive orders, are as follows:

First differences.

$b - c$ is $= .001,685,014,492$;
$c - d = .001,464,124,410$;
$d - e = .001,283,992,711$;
$e - f = .001,135,173,816$;
$f - g = .001,010,808,277$;
$g - h = .000,905,817,276$.

Second differences.

.000,220,890,082;
.000,180,131,699;
.000,148,818,895;
.000,124,365,539;
.000,104,991,001.

Third differences.

.000,040,758,383;
.000,031,312,804;
.000,024,453,356;
.000,019,374,538.

Fourth differences.

.000,009,445,579;
.000,006,859,448;
.000,005,078,818.

Fifth differences.

.000,002,586,131;
.000,001,780,630.

Sixth differences.

.000,000,805,501.

Therefore D^{I} is $= .001,685,014,492$;
$\text{D}^{\text{II}} = .000,220,890,082$;
$\text{D}^{\text{III}} = .000,040,758,383$;
$\text{D}^{\text{IV}} = .000,009,445,579$;
$\text{D}^{\text{V}} = .000,002,586,131$;
$\text{D}^{\text{VI}} = .000,000,805,501$.

Consequently the differential series

$$a - \frac{bx}{1+x} - \frac{\text{D}^{\text{I}}xx}{\overline{1+x}|^2} - \frac{\text{D}^{\text{II}}x^3}{\overline{1+x}|^3} - \frac{\text{D}^{\text{III}}x^4}{\overline{1+x}|^4} - \frac{\text{D}^{\text{IV}}x^5}{\overline{1+x}|^5} - \frac{\text{D}^{\text{V}}x^6}{\overline{1+x}|^6} - \frac{\text{D}^{\text{VI}}x^7}{\overline{1+x}|^7} - \text{\&c. is} = \text{to}$$

.025,

$$
\begin{aligned}
&.025,979,075,500,\\
-\ &.024,019,115,661, \times \frac{x}{1+x}\\
-\ &.001,685,014,492, \times \frac{xx}{\overline{1+x}|^2}\\
-\ &.000,220,890,082, \times \frac{x^3}{\overline{1+x}|^3}\\
-\ &.000,040,758,383, \times \frac{x^4}{\overline{1+x}|^4}\\
-\ &.000,009,445,579, \times \frac{x^5}{\overline{1+x}|^5}\\
-\ &.000,002,586,131, \times \frac{x^6}{\overline{1+x}|^6}\\
-\ &.000,000,805,501, \times \frac{x^7}{\overline{1+x}|^7}\\
-\ &\&c.
\end{aligned}
$$

This is the general value of the ſaid differential ſeries, whatever may be the value of x, or $\frac{vv}{ss}$. But in the caſe here ſuppoſed of an arch of 90°, the verſed ſine v is equal to the ſine s; and therefore $\frac{vv}{ss}$, or x, is $= 1$, and $\frac{x}{1+x} = \frac{1}{1+1}$, or $\frac{1}{2}$. Therefore the foregoing differential ſeries is in this caſe equal to

$$
\begin{aligned}
&.025,979,075,500,\\
-\ &.024,019,115,661, \times \frac{1}{2}\\
-\ &.001,685,014,492, \times \frac{1}{4}
\end{aligned}
$$

— .000,

$- .000,220,890,082, \times \frac{1}{8}$

$- .000,040,758,383, \times \frac{1}{16}$

$- .000,009,445,579, \times \frac{1}{32}$

$- .000,002,586,131, \times \frac{1}{64}$

$- .000,000,805,501, \times \frac{1}{128}$

$-$ &c. $= .025,979,075,500 - .012,009,557,830,$
$- .000,421,253,623,$
$- .000,027,611,260,$
$- .000,002,547,398,$
$- .000,000,295,174,$
$- .000,000,040,408,$
$- .000,000,006,292,$
$-$ &c.

$= .025,979,075,500, - .012,461,311,985,$ &c.
$= .013,517,763,515, -$ &c. Therefore the ſeries

$a - bx + cxx - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 +$ &c., or

$.025,979,075,500 - .024,019,115,661, x$
$+ .022,334,101,169, xx - .020,869,976,759, x^3$
$+ .019,585,984,048, x^4 - .018,450,810,232, x^5$
$+ .017,440,001,955, x^6 - .016,534,184,679, x^7$
$+$ &c. is in this caſe $= .013,517,763,515, -$ &c.
Therefore $\frac{v^{24}}{s^{24}} \times$ this laſt ſeries, or $\frac{v^{24}}{s^{24}} \times$ the ſeries

.025

$$.025,979,075,500, - .024,019,115,661, \frac{vv}{ss}$$
$$+ .022,334,101,169, \frac{v^4}{s^4} - .020,869,976,759, \frac{v^6}{s^6}$$
$$+ .019,585,984,048, \frac{v^8}{s^8} - .018,450,810,232, \frac{v^{10}}{s^{10}}$$
$$+ .017,440,001,955, \frac{v^{12}}{s^{12}} - .016,534,184,679, \frac{v^{14}}{s^{14}}$$

+ &c. is in this case $= \frac{v^{24}}{s^{24}} \times .013,517,763,515, -$ &c.; that is, the series

$$.025,979,075,500, \frac{v^{24}}{s^{24}} - .024,019,115,661, \frac{v^{26}}{s^{26}}$$
$$+ .022,334,101,169, \frac{v^{28}}{s^{28}} - .020,869,976,759, \frac{v^{30}}{s^{30}}$$
$$+ .019,585,984,048, \frac{v^{32}}{s^{32}} - .018,450,810,232, \times \frac{v^{34}}{s^{34}}$$
$$+ .017,440,001,955, \frac{v^{36}}{s^{36}} - .016,534,184,679, \frac{v^{38}}{s^{38}}$$

+ &c., or $\frac{Nv^{24}}{s^{24}} - \frac{Ov^{26}}{s^{26}} + \frac{Pv^{28}}{s^{28}} - \frac{Qv^{30}}{s^{30}} + \frac{Rv^{32}}{s^{32}} - \frac{Sv^{34}}{s^{34}} + \frac{Tv^{36}}{s^{36}} - \frac{Vv^{38}}{s^{38}} +$ &c. is in this case $= \frac{v^{24}}{s^{24}} \times .013,517,763,515, -$ &c. = (because v is in this case $= s$, and consequently $\frac{v^{24}}{s^{24}}$ is $= 1$) $.013,517,763,515, -$ &c. But we before found the value of the first twelve terms of the series $A - \frac{Bvv}{ss} + \frac{Cv^4}{s^4} - \frac{Dv^6}{s^6} + \frac{Ev^8}{s^8} - \frac{Fv^{10}}{s^{10}} +$ &c. to be in this case $=$ $.821,109,079,506$. Therefore the value of the whole series $A - \frac{Bvv}{ss} + \frac{Cv^4}{s^4} - \frac{Dv^6}{s^6} + \frac{Ev^8}{s^8} - \frac{Fv^{10}}{s^{10}} +$ &c. *ad infinitum* is in this case $= .821,109,079,506, + .013,517,763,515,$ — &c

– &c. = 834,626,843,021, – &c. of which the firſt eight figures .834,626,84 are exact, the error being in the ninth figure 3, which ought to be a 2 inſtead of a 3, as would have appeared if we had computed another term or two of the differential ſeries.

Art. 21. Since the ſeries $A - \frac{Bvv}{ss} + \frac{Cv^4}{s^4} - \frac{Dv^6}{s^6} + \frac{Ev^8}{s^8} - \frac{Fv^{10}}{s^{10}}$ + &c., or $1 - \frac{1.1.Avv}{2.2.ss} + \frac{3.3.Bv^4}{4.4.s^4} - \frac{5.5.Cv^6}{6.6.s^6} + \frac{7.7.Dv^8}{8.8.s^8} - \frac{9.9.Ev^{10}}{10.10.s^{10}}$ – &c. is, in this caſe of an arch of 90°, = .834,626,843, – &c., or ſomewhat leſs than .834,626,843, the product of that ſeries into $\pi \times \frac{rv}{s}$, will be $= \pi \times \frac{rv}{s} \times$ .834,626,843, – &c. = (becauſe v is in this caſe $= s$) $\pi \times r \times$ .834,626,843, – &c.

= 1.570,796,326,794, &c. $\times r \times$ .834,626,843 – &c. $= r \times$ 1.311,028,779, – &c., or ſomething leſs than $r \times$ 1.311,028,779; which is exact to nine places of figures, the more exact value of this quantity being $r \times$ 1.311,028,777,146, &c. as appears by a computation made by Mr. STIRLING, in his admirable Treatiſe on the Summation of Serieſes, p. 58.

Art. 22. This value of the product of 1.570,796, 326,794, &c. $\times \frac{rv}{s}$ into the ſeries $1 - \frac{1.1.Avv}{2.2.ss} + \frac{3.3.Bv^4}{4.4.s^4} - \frac{5.5.Cv^6}{6.6.s^6} + \frac{7.7.Dv^8}{8.8.s^8} - \frac{9.9.Ev^{10}}{10.10.s^{10}}$ + &c., found by the foregoing proceſſes in this extreme and moſt difficult caſe, to wit, 1.311,028,779, &c. $\times r$, exceeds its true

value,

value, 1.311,028,777,146, &c. × r by only .000,000,002, × r, or two thouſand-millionth parts of the radius r; which is indeed a moſt minute difference, and ſhews the great exactneſs and utility of this differential ſeries.

Art. 23. Of the nine figures to which the number 1.311,028,779, found by the foregoing proceſs, is exact, the laſt eight are owing to the differential ſeries. For if we were to multiply the value of the firſt twelve terms only of the ſeries $A - \frac{Bvv}{ss} + \frac{Cv^4}{s^4} - \frac{Dv^6}{s^6} + \frac{Ev^8}{s^8} - \frac{Fv^{10}}{s^{10}} +$ &c, or $1 - \frac{1.1.Avv}{2.2.ss} + \frac{3.3.Bv^4}{4.4.s^4} - \frac{5.5.Cv^6}{6.6.s^6} + \frac{7.7.Dv^8}{8.8.s^8} - \frac{9.9\,Ev^{10}}{10.10.s^{10}} +$ &c., to wit, the number .821,109,079,506, into $\pi \times r$, or 1.570,796,326,794, &c. × r, the product would be only 1.289, &c. × r, which is true to only one place of figures, the ſecond figure being a 2 inſtead of a 3. This therefore is an eminent proof of the utility of the ſaid differential ſeries.

Art. 24. In an arch of 90° the verſed ſine is equal to the radius of the circle, that is, according to the foregoing notation, v is $= r$. Therefore by art. 17. together with the foregoing computation, it appears, that the time of the deſcent of a pendulum, or other heavy body (moving freely from a ſtate of reſt by the force of gravity only) through the arch of a whole quadrant of a circle is to the time of the fall through the cor-

refpondent perpendicular altitude, or the radius, as 1.311,028,779, – &c. × *r* is to *r*, or as 1.311,028,779, – &c. is to 1.

Art. 25. Hence we may determine the proportion of the time of defcent of a pendulum through an arch of 90° to the time of its defcent through an infinitely fmall arch at the bottom of a quadrant, or rather (to fpeak correctly) to the limit of the time of defcent through a very fmall but finite arch at the bottom of the quadrant, to which the faid time continually approaches nearer and nearer as the faid fmall arch is taken lefs and lefs, and to which it may be made to approach fo nearly, by taking the faid fmall arch fufficiently fmall, as to differ from it by lefs than any given quantity. For this latter time, or limit, is known to be to the time of the fall of a heavy body through half the length of the pendulum, or half the radius of the circle, as the femi-circumference of a circle is to its diameter, that is, as the number 1.570,796,326,794, &c. is to 1. But the time of the fall of a heavy body through half the radius of the circle is to the time of the fall through the whole radius as 1 to $\sqrt{2}$, or 1.414,213, &c. Therefore, *ex æquo*, the faid limit of the time of defcent of a pendulum through a very fmall arch of the circle at the bottom of the quadrant, is to the time of the fall of a heavy body through

the radius of the circle, or the whole length of the pendulum, as 1.570,796, &c. is to 1.414,213, &c. But we have ſeen in the laſt article that the time of the fall of a heavy body through the radius of the circle is to the time of deſcent of a pendulum through the arch of a whole quadrant as 1 to 1.311,028,779, – &c. Therefore the limit of the time of deſcent of a pendulum through a very ſmall arch at the bottom of the quadrant is to the time of deſcent through the arch of the whole quadrant as 1.570,796,326, &c. × 1 is to 1.414,213, &c. × 1.311,028,779, – &c., or as 1.570,796,326, &c. is to 1.414,213, &c. × 1.311,028,779, – &c., that is, by art. 21. as 1.570,796,326, &c. is to 1.414,213, &c. × 1.570,796,326, &c. × .834,626,843, – &c., or as 1 to 1.414,213, &c. × .834,626,843, – &c., or as 1 to 1.180,340, &c., or, in ſmaller numbers, as 1 to 1.180, or as 1000 to 1180, or as 100 to 118, or as 50 to 59.

Art. 26. This proportion of the times of the deſcent of a pendulum through an infinitely ſmall arch at the bottom of the quadrant, and through the arch of the whole quadrant, agrees pretty nearly with that aſſigned for them by Mr. HUYGENS in the preface to his admirable Treatiſe on Pendulum-clocks, or *De Horologio Oſcillatorio*, which is that of 29 to 34. For 50 is to 59 as 29 is to 34.2, or $34\frac{1}{5}$; or (neglecting the fraction $\frac{1}{5}$) as 29 is to 34; Mr. HUYGENS meaning

meaning, probably, in that place, not to expreſs this proportion as accurately as he could, but only as nearly as it could be expreſſed by ſmall whole numbers. However, the numbers 50 and 59 expreſs this proportion rather more accurately than 29 and 34, and with pretty much the ſame degree of ſimplicity, and therefore, upon the whole, are ſomewhat to be preferred to them.

Art. 27. I have endeavoured to find another differential ſeries, ſimilar to that above deſcribed, for the purpoſe of inveſtigating the value of an infinite ſeries of this form, to wit, $a+bx+cxx+dx^3+ex^4+fx^5+gx^6+hx^7+$ &c. (in which all the terms are marked with the ſign +, or are added to the firſt term a) when the co-efficients b, c, d, e, f, g, h, &c. decreaſe very ſlowly, and x is very nearly equal to 1, and the terms of the ſeries decreaſe conſequently ſo ſlowly as to make the ſummation of it in the common way, or by the mere computation and addition of its terms, almoſt impracticable; but my endeavours have not been attended with ſucceſs. I may therefore, from my own experience, ſubſcribe to the truth of what is aſſerted upon this ſubject by the very learned and ingenious Mr. JAMES STIRLING in his Treatiſe, intitled, *Summatio Serierum*, p. 17. to wit, that *Series quarum termini ſunt per vices negativi et affirmativi, ſunt magis tractabiles quam alteræ, ubi de Summatione agitur*; though at firſt ſight one would be apt to imagine the reverſe of this propoſition to be true.